SET OPERATIONS

- *The essence of mathematics lies in its freedom.*
- *I see it but I don't believe it !*

– Georg Cantor

1.1 Introduction

The concept of a set is a base for all branches of mathematics. The theory of sets was developed by mathematician **Georg Cantor** (1845-1918 A.D). We have learnt some important and primary facts about sets in std. VIII.

In day-to-day life, we often talk about a group of same kind of objects; e.g., a herd of cows, a pack of cards, a team of players etc. This type of a **well-defined collection of objects is considered as a set.**

The father of set theory

The main inventor of set theory was the mathematician Georg Cantor. He was born on 3rd March, 1845 in St. Petesburg, Russia. He took his school education in St. Petesburg. In 1856, he moved to Germany. He was president of **Berlin Mathematical Society** (1864-1865). He achieved doctorate degree in 1867. He taught at a girls school in Berlin. In 1872, he was promoted as an extraordinary professor in Halle. He was a friend of Dedekind. He got some very surprising results in Mathematics. In 1873, he proved that rational numbers are countable. The birth of set theory dates to 1873, when Georg Cantor proved the uncountability of real line, actually December 7, 1873. Hilbert described Cantor's work as the finest product of mathematical genius and one of the supreme achievement of purely intellectual human activity. Some powerful people who disagreed with him severely criticized him for this. But today while those who troubled him are forgotten, Georg Cantor is remembered and widely respected. He died on 6th January, 1918 in Halle, Germany.

1.2 Important Points for Revision

- **A set is a well-defined collection of objects.**
- **A set without any member (element) is called a null set or an empty set.**
- **A set having only one member is called a singleton.**
- **$\in$ (belongs to) is an undefined symbol.**
- **If x is a member of the set A, we write $x \in A$**
- **If x is not a member of the set A, we write $x \notin A$.**
- **A set total number of members of which is a positive integer is called a finite set and a set which is not finite is called an infinite set. Null set is considered to be a finite set.**
- **If all the elements of a set A are present in the set B, then the set A is called a subset of the set B. This fact is denoted by $A \subset B$.**

Important points about subsets :

(1) **Empty set is a subset of every set. Thus, for any set A, $\emptyset \subset A$.**

(2) **Every set is a subset of itself. Thus, for any set A, $A \subset A$.**

(3) **If a set A has n elements, then number of its subsets is 2^n.**

(4) **$N \subset Z \subset Q \subset R$.**

- Generally, while dealing with a problem, we consider some definite set and its subsets. Such a definite set is called the **universal set** with reference to that problem. The **universal set** is denoted as U.

  A set which is a universal set for one problem need not be a universal set for another problem. For example, In Geometry, space or plane is a universal set. For interrelations of integers, set of integers Z is a universal set. For the solution of linear equations, the set of real numbers is a universal set.

- The set of all the elements which are in U but not in the given set A is called the **Complement of the set A**. It is denoted by A'.

  Thus, $A' = \{x \mid x \in U, x \notin A\}$

  so from the above definition, we get the following results.

  (1) $A \cup A' = U$ and (2) $A \cap A' = \emptyset$

- If two sets have same elements, they are said to be **equal sets. If every member of set A is a member of set B and every member of set B is a member of set A, then set A and set B are called equal sets. If A and B are equal sets we write A = B.** For equal sets A and B, $A \subset B$ and $B \subset A$.

i.e. if $A \subset B$ and $B \subset A$, then $A = B$.

For example let $A = \{x \mid x \in N, x < 5\}$ and $B = \{1, 2, 3, 4\}$ be two sets. Then both the sets A and B have the same members $\{1, 2, 3, 4\}$.

So, we say that $A = B$

- If every member of set A corresponds to one and only one member of set B and every member of set B corresponds to one and only one member of set A then the sets A and B are said to be in one-one correspondence with each other and the sets A and B are called **equivalent sets**. If set A is equivalent to set B, we write $A \sim B$.
- Thus, if two finite sets are in one-one correspondence with each other, then they should have the same number of elements.
- **Equal sets are always equivalent sets but equivalent sets need not be equal sets.**

For example, if $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ then $A \sim B$ but $A \neq B$.

But for infinite sets, situation is different.

If, $E = \{2, 4, 6, 8, ...\}$, then $N \sim E$. Because for every element of N, A unique number n is related to the number $2n$ belonging E and for every element of E, a unique number m is related to the number $\frac{m}{2} \in N$. But $E \subset N$.

EXERCISE 1.1

1. Classify the following sets in (a) as empty set or singleton set and in (b) as equal sets or equivalent sets :

(a) (1) $A = \{x \mid x \in Z, x + 1 = 0\}$

(2) $B = \{x \mid x \in N, x^2 - 1 = 0\}$

(3) $C = \{x \mid x \in N, x$ is a prime number between 13 and 17$\}$

(b) (1) $A = \{x \mid x \in N, x \leq 7\}$,

$B = \{x \mid x \in Z, -3 \leq x \leq 3\}$

(2) $A = \{x \mid x \in N, x$ is a multiple of $2, x < 10\}$,

$B = \{x \mid x \in N, x$ is an even natural number with a single digit$\}$

2. Find the number of subsets of the set $A = \{1, 2, 3\}$. Also write all the subsets of the set A.

3. If $A = \{x \mid x \in Z, x^2 - x = 0\}$, $B = \{x \mid x \in N, 1 \leq x \leq 4\}$, then can we say that $A \subset B$? Why ?

4. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 4, 6, 8\}$, then find A' and also verify that $A \cup A' = U$.

5. If A = {1, 2, 3}, B = {3, 4, 6}, then find all possible non-empty sets X which satisfy the following conditions :

(1) $X \subset A, X \not\subset B$ (2) $X \subset B, X \not\subset A$ (3) $X \subset A, X \subset B$

6. Examine whether the following statements are true or false :

(1) $\{1, 2, 3\} \subset \{1, 2, 3\}$ (2) $\{a, b\} \not\subset \{b, c, a\}$

(3) $\emptyset \notin \{\emptyset\}$ (4) $\{3\} \subset \{1, 2, \{3\}, 4\}$

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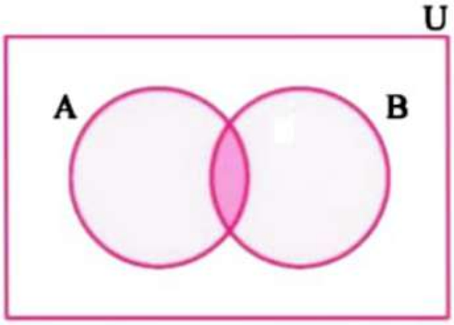
1.3 Properties of the Union Operation

Union set : For any two sets A and B, the set consisting of all the elements which are either in A or in B (or in both) is called the union set of the sets A and B and it is denoted by A ∪ B. The process of finding the union of two set is called the union operation.



Venn diagram of A ∪ B

Figure 1.1

Thus, in symbol, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. Venn-diagram is useful in understanding various relations between sets. In Venn-diagram 1.1 the coloured region describes A ∪ B.

Example 1 : Let A = {1, 3, 5, 7, 9}, B = {2, 4, 6, 8} be two sets. Find A ∪ B.

Solution : $A \cup B = \{1, 3, 5, 7, 9\} \cup \{2, 4, 6, 8\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Example 2 : If α = The letters of the word AHMEDABAD and β = The letters of the word BARODA are two sets, then find $\alpha \cup \beta$.

Solution : Here α = {A, B, D, E, H, M} and

β = {A, B, D, O, R}

$\therefore \alpha \cup \beta$ = {A, B, D, E, H, M} ∪ {A, B, D, O, R}

= {A, B, D, E, H, M, O, R}

Properties : Following are some rules followed by union operation. We will verify them with the help of illustrations.

(1) Union is a Binary Operation : For any two sets A and B, if $A \subset U$ and $B \subset U$, then $(A \cup B) \subset U$.

Suppose $U = \{x \mid x \in N, 1 \leq x \leq 6\}$, A = {1, 2, 3}, B = {2, 3, 4, 5}

Here, U = {1, 2, 3, 4, 5, 6}

So, $A \subset U$ and $B \subset U$

Now, $A \cup B = \{1, 2, 3\} \cup \{2, 3, 4, 5\}$

$= \{1, 2, 3, 4, 5\}$

Clearly, all members of $A \cup B$ are in U.

So, $(A \cup B) \subset U$. This result says union is a binary operation.

(2) Commutative Law : For any two sets A and B, $A \cup B = B \cup A$.

Let $A = \{c, d, e, f\}$, $B = \{p, q, r, s, t\}$ be any two sets.

Then,

$A \cup B = \{c, d, e, f\} \cup \{p, q, r, s, t\}$

$= \{c, d, e, f, p, q, r, s, t\}$ **(i)**

Now, $B \cup A = \{p, q, r, s, t\} \cup \{c, d, e, f\}$

$= \{c, d, e, f, p, q, r, s, t\}$ **(ii)**

Thus from (i) and (ii), $A \cup B$ and $B \cup A$ have the same elements.

Therefore, $A \cup B = B \cup A$

This law is known as commutative law for union, i.e. union is a commutative operation.

(3) Associative Law :

For any three sets A, B and C, $(A \cup B) \cup C = A \cup (B \cup C)$

Suppose $A = \{x \mid x \in N, 1 \leq x \leq 5\}$, $B = \{x \mid x \in N, x \text{ is an even number}, x < 10\}$,

$C = \{x \mid x \in N, x \text{ is a multiple of } 3, x < 10\}$

Now, $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 6, 9\}$ are given sets.

So, $A \cup B = \{1, 2, 3, 4, 5\} \cup \{2, 4, 6, 8\}$

$= \{1, 2, 3, 4, 5, 6, 8\}$

$\therefore \quad (A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8\} \cup \{3, 6, 9\}$

$= \{1, 2, 3, 4, 5, 6, 8, 9\}$ **(i)**

Now, $B \cup C = \{2, 4, 6, 8\} \cup \{3, 6, 9\}$

$= \{2, 3, 4, 6, 8, 9\}$

$\therefore \quad A \cup (B \cup C) = \{1, 2, 3, 4, 5\} \cup \{2, 3, 4, 6, 8, 9\}$

$= \{1, 2, 3, 4, 5, 6, 8, 9\}$ **(ii)**

$\therefore$ By results (i) and (ii) we verify that union is associative.

From Venn-diagram 1.2 it can be verified that,

$(A \cup B) \cup C = A \cup (B \cup C)$

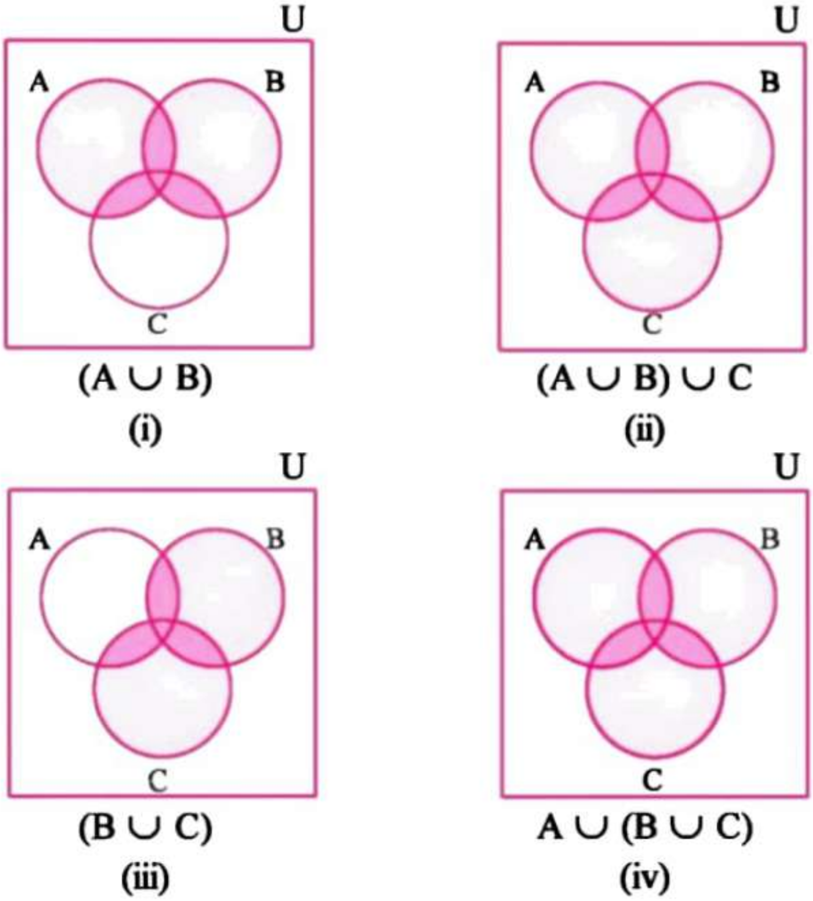


Figure 1.2

In Venn diagram 1.2 coloured region describes the set mentioned below the Venn-diagram.

This law is known as the associative law for union.

i.e. **union is an associative operation.**

(4) For any two sets A and B, $A \subset (A \cup B)$ and $B \subset (A \cup B)$

Let $A = \{x \mid x \in Z, x^2 - 4 = 0\}$, $B = \{x \mid x \in N, x \leq 5\}$ be two sets.

$\therefore$ Here $A = \{-2, 2\}$, $B = \{1, 2, 3, 4, 5\}$

Now $A \cup B = \{-2, 2\} \cup \{1, 2, 3, 4, 5\}$

$= \{-2, 1, 2, 3, 4, 5\}$

Clearly $A \subset (A \cup B)$ and $B \subset (A \cup B)$.

Look at the Venn-diagram 1.3. Here A and B are two sets. The set A consists of the regions R_1 and R_2; the set B consists of the regions R_2 and R_3. So, $A \cup B$ consists of the regions R_1, R_2 and R_3. Thus, the regions R_1 and R_2 are included in the regions R_1, R_2 and R_3 together. i.e. the set A is contained in the set $A \cup B$, so $A \subset (A \cup B)$.

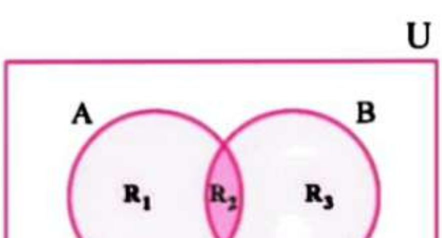


Figure 1.3

Similarly the regions R_2 and R_3 are included in the regions R_1, R_2 and R_3 together. That means the set B is contained in the set $A \cup B$. i.e. $B \subset (A \cup B)$

In general, **for any two sets A and B, $A \subset (A \cup B)$ and $B \subset (A \cup B)$.**

(5) If $A \subset B$, then $A \cup B = B$

Let us try to understand the above property by following example.

Example 3 : If α = The set of the letters of the word GATE and β = The set of the letters of the word LOCGATE are two sets, then verify that $\alpha \subset \beta$ and $\alpha \cup \beta = \beta$

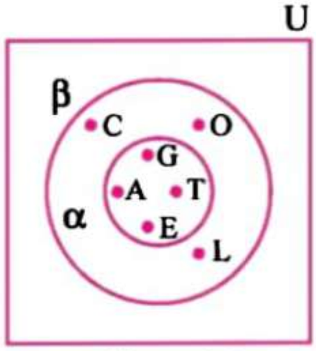


Figure 1.4

Solution : Here $\alpha = \{G, A, T, E\}$,

$$\beta = \{L, O, C, G, A, T, E\}$$

$\therefore$ Here all the elements of the set α are present in β.

$\therefore \quad \alpha \subset \beta$

Now, $\alpha \cup \beta = \{G, A, T, E\} \cup \{L, O, C, G, A, T, E\}$

$= \{L, O, C, G, A, T, E\}$

$\therefore \quad \alpha \cup \beta = \beta$

(6) $A \cup U = U$ and $A \cup \emptyset = A$

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set and

$A = \{x \mid x \in N, x \text{ is a prime number less than } 10\}$

be a given set

$\therefore \quad A = \{2, 3, 5, 7\}$

Thus, $A \cup U = \{2, 3, 5, 7\} \cup \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = U$

$A \cup \emptyset = \{2, 3, 5, 7\} \cup \emptyset$

$= \{2, 3, 5, 7\}$

$= A$

Thus, we say that $A \cup U = U$ and $A \cup \emptyset = A$.

1.4 Properties of the Intersection Operation

Now we will study some rules about operation of intersection and verify them with the help of illustrations.

Intersection set : For any two sets A and B, the set consisting of all the elements which belong to both the sets A and B is called the intersection set of two sets A and B and it is denoted by $A \cap B$.

Thus, in symbols $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

In Venn-diagram 1.5 the coloured region describes $A \cap B$.

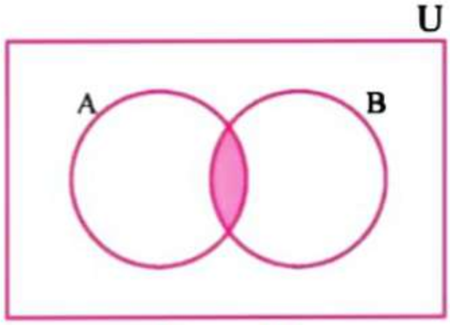


Figure 1.5

Example 4 : Let $A = \{x \mid x \in N, x \text{ is a multiple of } 3, x \leq 15\}$

$B = \{x \mid x \in Z, 0 < x < 10\}$ be two sets.

Find $A \cap B$.

Solution :

Here $A = \{x \mid x \in N, x \text{ is a multiple of } 3, x \leq 15\}$

$= \{3, 6, 9, 12, 15\}$

$B = \{x \in Z, 0 < x < 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\therefore\ A \cap B = \{3, 6, 9, 12, 15\} \cap \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \{3, 6, 9\}$

Let us verify some properties of intersection by examples.

Properties :

(1) Intersection is a Binary Operation : For two sets A and B, if $A \subset U$, $B \subset U$, then $(A \cap B) \subset U$.

Suppose $U = \{x \mid x \in N, 1 \leq x \leq 25\}$, $A = \{1, 4, 9, 16, 25\}$

$B = \{4, 8, 12, 16, 20\}$

Since $U = \{1, 2, 3, ..., 25\}$, $A \subset U$, $B \subset U$

Now $A \cap B = \{1, 4, 9, 16, 25\} \cap \{4, 8, 12, 16, 20\} = \{4, 16\}$

Clearly, each member of $A \cap B$ is in U.

$\therefore\ (A \cap B) \subset U$

So, intersection is a binary operation.

(2) Commutative Law: For any two sets A and B, $A \cap B = B \cap A$

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 6, 8\}$ be any two sets.

Then $A \cap B = \{1, 2, 3, 4, 5\} \cap \{3, 4, 6, 8\} = \{3, 4\}$ **(i)**

and $B \cap A = \{3, 4, 6, 8\} \cap \{1, 2, 3, 4, 5\}$

$= \{3, 4\}$ **(ii)**

Thus, $A \cap B = B \cap A$ **(from (i) and (ii))**

This law is known as commutative law for intersection i.e. intersection is a commutative operation.

(3) Associative Law : For any three sets A, B and C,

$(A \cap B) \cap C = A \cap (B \cap C)$

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6\}$, $C = \{1, 4, 7\}$

$\therefore\ A \cap B = \{3, 4, 5\}$

$\therefore\ (A \cap B) \cap C = \{4\}$ (i)

$B \cap C = \{4\}$

$\therefore\ A \cap (B \cap C) = \{4\}$ (ii)

Thus, $(A \cap B) \cap C = A \cap (B \cap C)$ **((i) and (ii))**

Now, Let us verify the law with the help of Venn-diagram.

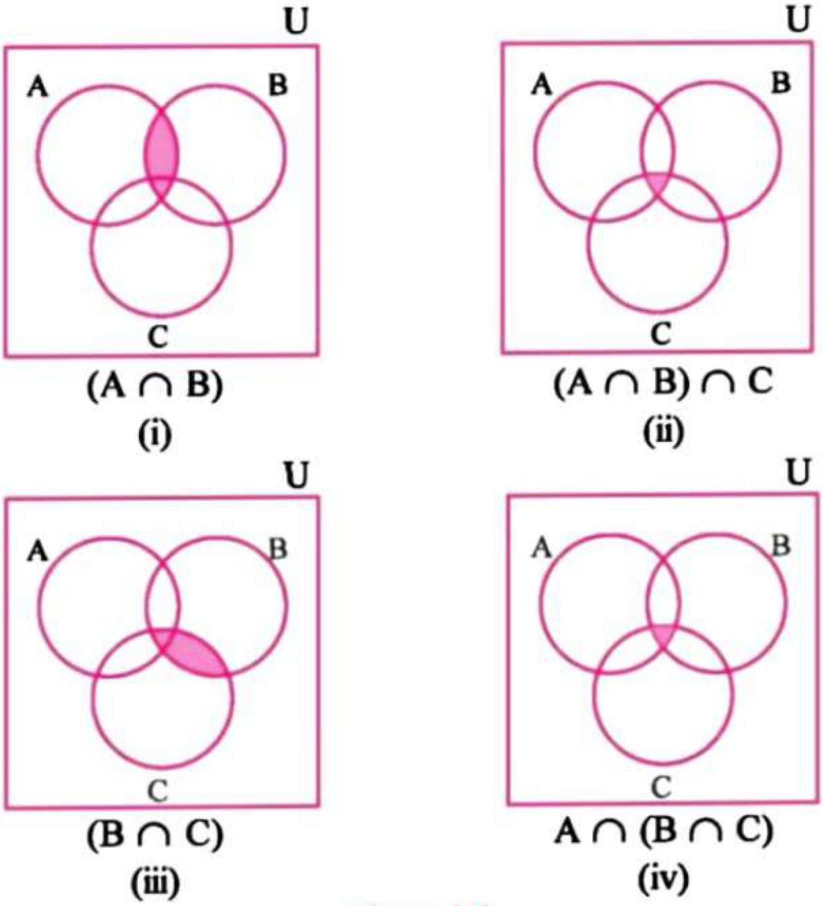


Figure 1.6

In Venn-diagram 1.6, coloured region describes the set mentioned below the Venn-diagram.

It can be seen from the Venn-diagram 1.6 that, $(A \cap B) \cap C = A \cap (B \cap C)$

In general, **for any three sets A, B, and C, $(A \cap B) \cap C = A \cap (B \cap C)$**

This rule is known as the associative law for the operation of intersection.

(4) $(A \cap B) \subset A$ and $(A \cap B) \subset B$.

All the elements of $A \cap B$ belong to the sets A and B.

Hence $(A \cap B) \subset A$ and $(A \cap B) \subset B$

Look at the Venn-diagram 1.7. The set A consists of the regions R_1 and R_2. The set B consists of the regions R_2 and R_3. The region R_2 is common to both A and B. Thus, $A \cap B$ consists of the region R_2. The region R_2 is contained in A as well as in B. Thus, $(A \cap B) \subset A$ and $(A \cap B) \subset B$.

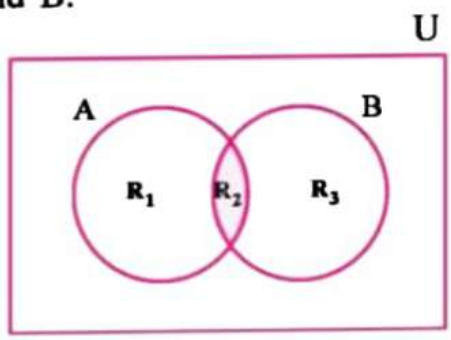


Figure 1.7

(5) If $A \subset B$, then $A \cap B = A$

Let us verify this property by an example.

Example 5 : Let $A = \{x \mid x \in N, x^2 - 9 = 0\}$ and $B = \{x \mid x \in N, x < 5\}$ be two given sets. Verify that $A \subset B$ and $A \cap B = A$

Solution : Here $x^2 - 9 = 0$

$\therefore \quad x^2 - 3^2 = 0$

$\therefore \quad (x + 3)(x - 3) = 0$

$\therefore \quad x = -3 \quad$ or $\quad x = 3$

as $x \in N$, $x = -3$ is not possible.

$\therefore \quad x = 3$

Hence $A = \{3\}$ **(i)**

Now, $B = \{x \mid x \in N, x < 5\}$

$= \{1, 2, 3, 4\}$ **(ii)**

Hence by the results (i) and (ii), we can say that $A \subset B$.

Now, $A \cap B = \{3\} \cap \{1, 2, 3, 4\} = \{3\} = A$

$\therefore$ If $A \subset B$, then $A \cap B = A$

Similarly If $B \subset A$, then $A \cap B = B$. (Verify it by yourself)

(6) $A \cap \emptyset = \emptyset$ and $A \cap U = A$

Let $U = \{1, 2, 3, 4, 5\}$, $A = \{2, 3\}$

Then obviously $A \cap U = \{2, 3\} = A$ and $A \cap \emptyset = \emptyset$

Disjoint sets : For any two non-empty sets A and B, if $A \cap B = \emptyset$ then the sets A and B are said to be disjoint

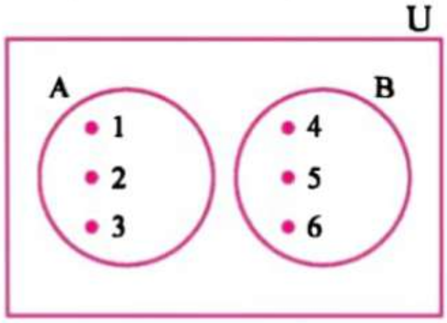
Example 6 : If $A = \{1, 2, 3\}$, $B = \{x \mid x \in N, 3 < x < 7\}$ are two sets, are they disjoint ?

Solution : Here $B = \{4, 5, 6\}$

Hence $A \cap B = \{1, 2, 3\} \cap \{4, 5, 6\} = \emptyset$

So there is no element common to both A and B. Hence we say that A and B are disjoint sets.

By Venn-diagram 1.8 we can also understand the above definition very easily.



$A \cap B = \emptyset$

Figure 1.8

1.5 Distributive Laws

We are familiar with the distributive law of multiplication over addition for real numbers.

For all $a, b, c \in R$, $a \times (b + c) = (a \times b) + (a \times c)$

For example if $a = 3$, $b = 4$, $c = 5$, then

L.H.S. $= a \times (b + c) = 3 \times (4 + 5) = 3 \times 9 = 27$

R.H.S. $= (a \times b) + (a \times c) = (3 \times 4) + (3 \times 5) = 12 + 15 = 27$

$\therefore$ L.H.S. = R.H.S.